SEAMO 2020 PAPER F

Problems and Solutions

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Written with LaTeX

QUESTIONS 1 TO 10 ARE WORTH 3 MARKS EACH

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1. Find the last two digits of 11^{2020}.

   A. 01
   B. 41
   C. 71
   D. 91
   E. None of the above

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2. The quadratic equation

$$x^2 - 52x + k = 0$$

   has roots that are prime numbers. Find the maximum value of k.

   A. 520
   B. 576
   C. 640
   D. 667
   E. None of the above

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3. Let $f(x) = x^2 + 2020x + 20$.
   How many ordered pairs of positive integers (m, n) are there such that $f(m+n) = f(m) + f(n)$?

   A. 1
   B. 2
   C. 3
   D. 4
   E. None of the above

---

4. Find the sum of all possible positive integers n such that the expression below is an integer.

$$\frac{4n^3 - 16n^2 + 29n + 60}{2n - 3}$$

   A. 42
   B. 69
   C. 75
   D. 81
   E. None of the above

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5. Evaluate the sum

$$S = \frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \cdots + \frac{20}{1+20^2+20^4}$$

A. $\frac{1}{2}$
B. $\frac{210}{421}$
C. $\frac{105}{211}$
D. 1
E. None of the above

---

6. 2 white, 3 black and 4 grey marbles are shared equally among 9 students.

Find the number of ways the marbles can be distributed so that Bran and Sansa gets the same colour and Arya gets a grey marble.

A. 120
B. 130
C. 140
D. 150
E. None of the above

---

7. Given that a is a real number such that $a^4 + a^3 + a^2 + a + 1 = 0$.

Evaluate $a^{2020} + 2a^{2010} + 3a^{2000}$.

A. 2
B. 4
C. 6
D. 8
E. None of the above

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8. Given that a, b, and c are three distinct real numbers such that

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}.$$

What is the largest possible value of abc?

A. $\frac{1}{2}$
B. 2
C. $\frac{5}{2}$
D. 3
E. None of the above

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9. How many positive integers less that 2020 with the property that the sum of it digits equals 9?

   A. 50
   B. 100
   C. 102
   D. 202
   E. None of the above

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10. The sequence a_n is defined by

$$a_{n+2} = \frac{1 + a_{n+1}}{a_n}$$

with $a_1 = 1$ and $a_2 = 2$. Evaluate a_{2020}.

   A. 1
   B. 2
   C. 3
   D. 4
   E. None of the above

QUESTIONS 11 TO 20 ARE WORTH 4 MARKS EACH

11. Find the smallest prime factor of

$$\underbrace{1000\cdots01}_{\text{2020 zeros}}$$

A. 3
B. 5
C. 7
D. 11
E. None of the above

12. In the expansion of

$$f(x) = (1+ax)^4(1+bx)^5$$

where a and b are positive integers, the coefficient of x^2 is 86. Evaluate $a+b$.

A. 2
B. 3
C. 4
D. 5
E. None of the above

13. The equation $x^3 - ax^2 + bx - 2020$ has three positive integer roots.

Find the least possible value of a.

A. 101
B. 110
C. 202
D. 220
E. None of the above

14. Evaluate the sum

$$S = \sin^2 0° + \sin^2 2° + \sin^2 4° \cdots + \sin^2 180°$$

.

A. 80
B. 81
C. 88
D. 90
E. None of the above

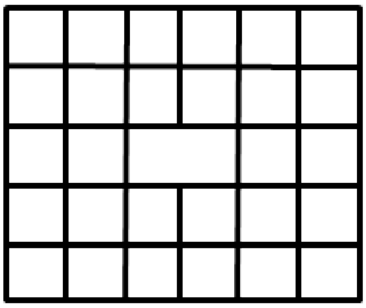
15. Given that a and b are real numbers satsifying

$$\begin{cases} 6 - 5a + 4b - 3a^2 + 2ab - b^2 = 0, \\ a - b = 1. \end{cases}$$

Find the sum of all possible values of $\dfrac{30a}{b}$.

A. -15
B. -10
C. 15
D. 30
E. None of the above

16. The figure below shows a 5×6 rectangular board with a missing 1×2 rectangle in the center.



How many squares are there in the board?

A. 14
B. 30
C. 54
D. 56
E. None of the above

17. In ΔABC,

$$(\sin A + \sin B) : (\sin B + \sin C) : (\sin C + \sin A) = 19\ : 20\ : 21.$$

Find the value of $99 \cos A$.

A. 39
B. 41
C. 51
D. 60
E. None of the above

18. Find the least positive integer n such that the equation $\left\lfloor \frac{10^n}{x} \right\rfloor = 98$ has integer solution x.

$\lfloor k \rfloor$ is the largest integer smaller than or equal to k.

A. 3
B. 4
C. 5
D. 6
E. None of the above

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19. How many positive integers $k < 100$ such that

$$2(5^{6n}) + k(2^{3n+2}) - 1$$

is divisible by 7 for any positive integer n?

A. 12
B. 14
C. 18
D. 19
E. None of the above

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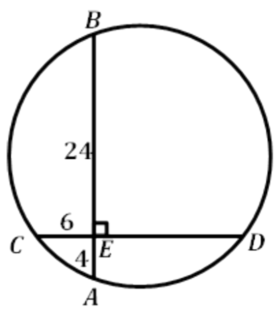
20. A, B, C, and D are four distinct points lying on the circumference of a circle such that chords AB and CD are perpendicular at point E.



Given that $EA = 4$, $EB = 2$, and $EC = 6$, find the radius of the circle.

A. $\sqrt{221}$
B. 15
C. $\sqrt{270}$
D. 18
E. None of the above

QUESTIONS 21 TO 25 ARE WORTH 6 MARKS EACH

21. You need to tile a 10×1 hallway with a supply of 1×1 red, 2×1 red tiles and 2×1 blue tiles. find the number of ways you can tile the 10×1 hallway.

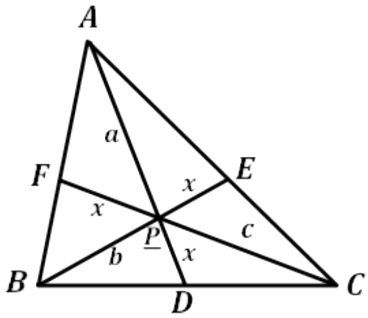
22. x, y, and z are real numbers such that

$$x + y + z = 7$$
$$x^2 + y^2 + z^2 = 19$$
$$x^3 + y^3 + z^3 = 64.$$

Evaluate $x^4 + y^4 + z^4$.

23. In ΔABC shown below, AD, BE, and CF intersect at P. Suppose $AP = a, BP = b, CP = c$ and $DP = EP = FP = x$.

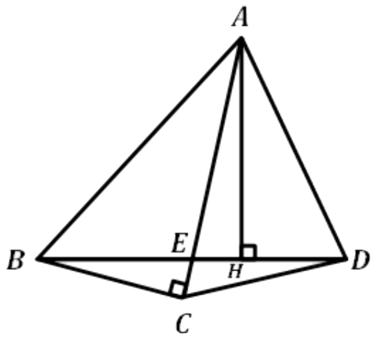
Given that $x = 3$ and $a + b + c = 20$, find abc.



24. Positive integers a, b, and c are randomly selected from the set $\{1, 2, 3, \cdots, 2020\}$ with replacement.

Find the probablility that $abc + ab + 2a$ is divisible by 5.

25. $ABCD$ is a convex quadrilateral such that AC intersects BD at E. H is a point lying in the segment DE such that AH is perpendicular to DE.
Suppose $BE = ED$, $CE = 9, EH = 12, AH = 32$, and $\angle BCA = 90°$. Evaluate the length of CD.



END OF PAPER

1. Find the last two digits of 11^{2020}.

   A. 01
   B. 41
   C. 71
   D. 91
   E. None of the above

English

Answer. 01

Solution.
Finding the last two digits of a number is equivalent to computing the remainder of that number when divided by 100. We investigate modulo 100 of 11^{2020}.
Note that $11^{10} \equiv 1 \pmod{100}$. Thus, we can rewrite 11^{2020} using congruence relation,

$$\begin{aligned} 11^{2020} = (11^{10})^{202} &\equiv 1^{202} \pmod{100} \\ &\equiv 1 \pmod{100}. \end{aligned}$$

Therefore, the last two digits of 11^{2020} is $\boxed{\textbf{A. 01}}$.

Bahasa Indonesia

Jawaban. 01

Solusi.
Mencari dua digit suatu bilangan sama saja dengan mencari sisa pembagian bilangan tersebut dengan 100. Kita tinjau modulo 100 dari 11^{2020}.
Perhatikan bahwa $11^{10} \equiv 1 \pmod{100}$ sehingga

$$11^{2020} = (11^{10})^{202} \equiv (1)^{202} \equiv 1 \pmod{100},$$

berdasarkan sifat modular aritmetik.
Dengan demikian, dua digit terakhir dari 11^{2020} adalah $\boxed{\textbf{A. 01}}$.

2. The quadratic equation

$$x^2 - 52x + k = 0$$

has roots that are prime numbers. Find the maximum value of k.

A. 520

B. 576

C. 640

D. 667

E. None of the above

English

Answer. 667

Solution.
Let x_1 and x_2 be the roots of equation $x^2 - 52x + k = 0$. The relationship between roots of a polynomial and its coefficients are obtained from Vieta's formula. Hence we have

$$x_1 + x_2 = 52 \text{ and } x_1 x_2 = k.$$

Our objective is equivalent to calculating the maximum of the product of the two roots. Because x_1 and x_2 are both primes, we can feasibly list primes less than 52:

$$\{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}.$$

Ordered pairs satisfying $x_1 + x_2 = 52$ are (5, 47), (11, 41), and (23, 29). Hence, the maximum value of k is $\boxed{\textbf{D. 667}}$.

Bahasa Indonesia

Jawaban. 667

Solusi.
Misalkan x_1 dan x_2 adalah akar-akar dari persamaan $x^2 - 52x + k = 0$. Hubungan antara akar-akar polinomial dengan koefisiennya didapatkan dari rumus Vieta. Maka, kita punya

$$x_1 + x_2 = 52 \text{ dan } x_1 x_2 = k.$$

Untuk menjawab persoalan, maka kita cukup mencari maksimum dari perkalian kedua akar. Karena x_1 dan x_2 keduanya prima, kita dapat mendaftarkan bilangan prima yang kurang dari 52:

$$\{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}.$$

Pasangan bilangan yang memenuhi adalah (5, 47), (11, 41), dan (23, 29). Hasil kali terbesar didapatkan ketika $x_1 = 23$ dan $x_2 = 29$. Maka, solusi dari persoalan kali ini adalah $\boxed{\textbf{D. 667}}$.

3. Let

$$f(x) = x^2 + 2020x + 20.$$

How many ordered pairs of positive integers (m, n) are there such that $f(m+n) = f(m) + f(n)$?

A. 1

B. 2

C. 3

D. 4

E. None of the above

English

Answer. 4

Solution.
Consider the equation $f(m+n) = f(m) + f(n)$:

$$\begin{aligned} f(m+n) &= f(m) + f(n) \\ (m+n)^2 + 2020(m+n) + 20 &= m^2 + 2020m + 20 + n^2 + 2020n + 20 \\ \implies mn &= 10. \end{aligned}$$

Ordered pairs of positive integers (m, n) that satisfy the above condition are

$$\{(1, 10), (2, 5), (5, 2), (10, 1)\}.$$

Therefore, there are $\boxed{\textbf{D. 4}}$ ordered pairs of positive integers (m, n) that satisfy $f(m+n) = f(m) + f(n)$.

Bahasa Indonesia

Jawaban. 4

Solusi.
Tinjau persamaan $f(m+n) = f(m) + f(n)$:

$$\begin{aligned} f(m+n) &= f(m) + f(n) \\ (m+n)^2 + 2020(m+n) + 20 &= m^2 + 2020m + 20 + n^2 + 2020n + 20 \\ \implies mn &= 10. \end{aligned}$$

Pasangan bilangan bulat positif (m, n) yang memenuhi kondisi tersebut adalah

$$\{(1, 10), (2, 5), (5, 2), (10, 1)\}.$$

Dengan demikian, terdapat $\boxed{\textbf{D. 4}}$ pasangan solusi bulat positif yang memenuhi persamaan $f(m+n) = f(m) + f(n)$.

4. Find the sum of all possible positive integers n such that the expression below is an integer.

$$\frac{4n^3 - 16n^2 + 29n + 60}{2n - 3}$$

A. 42

B. 69

C. 75

D. 81

E. None of the above

English

Answer. 69

Solution.
In order to examine all possible n, we modify the expression to the following:

$$\frac{4n^3 - 16n^2 + 29n + 60}{2n - 3} = 2n^2 - 5n + 7 + \frac{81}{2n - 3}.$$

Notice that the first three terms are integers. In order for the entire expression to be an integer, $\frac{81}{2n-3}$ must be an integer. Hence, $(2n - 3)$ must be a factor of 81, which corresponds to

$$2n - 3 = x,\ x \in \{\pm 1, \pm 3, \pm 9, \pm 27, \pm 81\}.$$

Evaluate each n for each x, pick every $n > 0$, and summing all possible values of n, we have $1 + 2 + 3 + 6 + 15 + 42 =$ $\boxed{\textbf{B. 69}}$ as our final answer.

Bahasa Indonesia

Jawaban. 69

Solusi.
Kita modifikasi ekspresi di atas dengan memisahkan bagian bulat dan bagian pecahan:

$$\frac{4n^3 - 16n^2 + 29n + 60}{2n - 3} = 2n^2 - 5n + 7 + \frac{81}{2n - 3}.$$

Ketiga suku pertama merupakan bilangan bulat. Supaya ekspresi awal berbentuk bilangan bulat, $\frac{81}{2n-3}$ haruslah bulat. Maka, $(2n - 3)$ pasti merupakan faktor dari 81:

$$2n - 3 = x,\ x \in \{\pm 1, \pm 3, \pm 9, \pm 27, \pm 81\}.$$

Hitung tiap n untuk tiap x, ambil seluruh $n > 0$, dan jumlahkan semua n yang memenuhi, jawaban akhir kita untuk persoalan kali ini adalah $1 + 2 + 3 + 6 + 15 + 42 =$ $\boxed{\textbf{B. 69}}$.

5. Evaluate the sum

$$S = \frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \cdots + \frac{20}{1+20^2+20^4}$$

A. $\frac{1}{2}$

B. $\frac{210}{421}$

C. $\frac{105}{211}$

D. 1

E. None of the above

English

Answer. $\frac{210}{421}$

Solution.
First of all, notice that

$$\begin{aligned} n^6 - 1 &= (n^2-1)(n^4+n^2+1) \\ (n^3-1)(n^3+1) &= (n^2-1)(n^4+n^2+1) \\ (n^2-1)(n^2+n+1)(n^2-n+1) &= (n^2-1)(n^4+n^2+1). \\ \implies \frac{n}{n^4+n^2+1} &= \frac{n}{(n^2-n+1)(n^2+n+1)}. \end{aligned}$$

The expression above may be rewritten as following:

$$\frac{n}{(n^2-n+1)(n^2+n+1)} = \frac{1}{2}\left(\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right).$$

Let

$$f(n) = \frac{n}{n^4+n^2+1} = \frac{1}{2}\left(\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right).$$

Therefore, S can be expressed as:

$$\begin{aligned} S &= f(1) + f(2) + \cdots + f(20) \\ S &= \frac{1}{2}\left(\frac{1}{1^2-1+1} - \frac{1}{1^2+1+1}\right) + \cdots + \frac{1}{2}\left(\frac{1}{20^2-20+1} - \frac{1}{20^2+20+1}\right) \end{aligned}$$

The terms above form telescoping series[a] that cancel each other. Hence,

$$S = \frac{1}{2}\left(\frac{1}{1^2-1+1} - \frac{1}{20^2+20+1}\right) = \frac{210}{421}.$$

Finally, S equals $\boxed{\textbf{B.}\ \frac{210}{421}}$.

[a] The second term of $f(n)$ and the first term of $f(n+1)$ for $n = 1, 2, \cdots 19$ cancel each other.

Bahasa Indonesia

Jawaban. $\dfrac{210}{421}$

Solusi.
Pertama-tama, perhatikan bahwa

$$\begin{aligned} n^6 - 1 &= (n^2-1)(n^4+n^2+1) \\ (n^3-1)(n^3+1) &= (n^2-1)(n^4+n^2+1) \\ (n^2-1)(n^2+n+1)(n^2-n+1) &= (n^2-1)(n^4+n^2+1). \\ \implies \frac{n}{n^4+n^2+1} &= \frac{n}{(n^2-n+1)(n^2+n+1)}. \end{aligned}$$

Perhatikan pula bahwa ekspresi di atas dapat ditulis ulang sebagai berikut:

$$\frac{n}{(n^2-n+1)(n^2+n+1)} = \frac{1}{2}\left(\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right).$$

Misalkan

$$f(n) = \frac{n}{n^4+n^2+1} = \frac{1}{2}\left(\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right).$$

Dengan demikian, S dapat dinyatakan ulang sebagai:

$$\begin{aligned} S &= f(1) + f(2) + \cdots + f(20) \\ S &= \frac{1}{2}\left(\frac{1}{1^2-1+1} - \frac{1}{1^2+1+1}\right) + \cdots + \frac{1}{2}\left(\frac{1}{20^2-20+1} - \frac{1}{20^2+20+1}\right) \end{aligned}$$

Suku-suku pada penjumlahan di atas membentuk barisan teleskoping[a] yang saling menghilangkan satu sama lain sehingga kita memiliki

$$S = \frac{1}{2}\left(\frac{1}{1^2-1+1} - \frac{1}{20^2+20+1}\right) = \frac{210}{421}.$$

Akhirnya, S bernilai $\boxed{\textbf{B. } \dfrac{210}{421}}$.

[a] Suku kedua dari $f(n)$ dan suku pertama dari $f(n+1)$ untuk $n = 1, 2, \cdots 19$ saling menghilangkan.

6. 2 white, 3 black and 4 grey marbles are shared equally among 9 students.

   Find the number of ways the marbles can be distributed so that Bran and Sansa gets the same colour and Arya gets a grey marble.

   A. 120
   B. 130
   C. 140
   D. 150
   E. None of the above

English

Answer. 140

Solution.
We count the number of ways into three cases separately.

First case: Both Bran and Sansa gets a white marble.
The rest of the students will get 3 black marbles and 3 grey marbles. The number of ways the marbles can be distributed is

$$\frac{6!}{3! \cdot 3!} = 20 \text{ ways.}$$

Second case: Both Bran and Sansa gets a black marble.
The rest of the students will get 2 white marbles, 1 black marble, and 3 grey marbles. The number of ways the marbles can be distributed is

$$\frac{6!}{2! \cdot 3!} = 60 \text{ ways.}$$

Third case: Both Bran and Sansa gets a grey marble.
The rest of the students will get 2 white marbles, 3 black marbles, and 1 grey marble. The number of ways the marbles can be distributed is

$$\frac{6!}{2! \cdot 3!} = 60 \text{ ways.}$$

Hence, the number of ways the marble can be distributed as required is $20 + 20 + 60 =$ **C. 140** ways.

Bahasa Indonesia

Jawaban. 140

Solusi.
Kita hitung banyaknya cara membagi kelereng dalam tiga kasus.

Kasus pertama: Bran dan Sansa mendapat kelereng putih.
Siswa lainnya akan mendapat 3 kelereng hitam dan 3 kelereng abu-abu. Banyaknya cara membagi kelereng tersebut adalah

$$\frac{6!}{3! \cdot 3!} = 20 \text{ cara.}$$

Kasus kedua: Bran dan Sansa mendapat kelereng hitam.
Siswa lainnya akan mendapat 2 kelereng putih, 1 kelereng hitam, dan 3 kelereng abu-abu. Banyaknya cara membagi kelereng tersebut adalah

$$\frac{6!}{2! \cdot 3!} = 60 \text{ cara.}$$

Kasus ketiga: Bran dan Sansa mendapat kelereng abu-abu.
Siswa lainnya akan mendapat 2 kelereng putih, 3 kelereng hitam, dan 1 kelereng abu-abu. Banyaknya cara membagi kelereng tersebut adalah

$$\frac{6!}{2! \cdot 3!} = 60 \text{ cara.}$$

Maka, banyaknya cara membagi kelereng dengan syarat demikian adalah $20 + 20 + 60 =$ $\boxed{\textbf{C. 140}}$ cara.

7. Given that a is a real number such that $a^4 + a^3 + a^2 + a + 1 = 0$.
Evaluate $a^{2020} + 2a^{2010} + 3a^{2000}$.

A. 2
B. 4
C. 6
D. 8
E. None of the above

English

Answer. 6

Solution.
Notice that
$$(a^5 - 1) = (a - 1)(a^4 + a^3 + a^2 + a + 1).$$
Since $a^4 + a^3 + a^2 + a + 1 = 0$, we have
$$a^5 - 1 = 0 \implies a^5 = 1.$$
Therefore,
$$\begin{aligned} a^{2020} + 2a^{2010} + 3a^{2000} &= (a^5)^{404} + 2(a^5)^{402} + 3(a^5)^{400} \\ &= 1^{404} + 2(1)^{402} + 3(1)^{400} = 6. \end{aligned}$$
Hence, $a^{2020} + 2a^{2010} + 3a^{2000}$ is $\boxed{\textbf{C. 6}}$

Bahasa Indonesia

Jawaban. 6

Solusi.
Perhatikan bahwa
$$(a^5 - 1) = (a - 1)(a^4 + a^3 + a^2 + a + 1).$$
Mengingat $a^4 + a^3 + a^2 + a + 1 = 0$, kita memiliki
$$a^5 - 1 = 0 \implies a^5 = 1.$$
Dengan demikian,
$$\begin{aligned} a^{2020} + 2a^{2010} + 3a^{2000} &= (a^5)^{404} + 2(a^5)^{402} + 3(a^5)^{400} \\ &= 1^{404} + 2(1)^{402} + 3(1)^{400} = 6. \end{aligned}$$
Akhirnya, $a^{2020} + 2a^{2010} + 3a^{2000}$ bernilai $\boxed{\textbf{C. 6}}$

8. Given that a, b, and c are three distinct real numbers such that

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}.$$

What is the largest possible value of abc?

A. $\frac{1}{2}$
B. 2
C. $\frac{5}{2}$
D. 3
E. None of the above

English

Answer. 1

Solution.
System of equation that involves symmetry has a unique property that usually we could exploit the cyclicity to obtain other equation(s). In this case, we subtract the leftmost by the middle expression, subtract the middle by the rightmost, and subtract the rightmost by the leftmost to obtain three equations:

$$\begin{aligned} a - b &= \frac{1}{c} - \frac{1}{b} = \frac{b-c}{bc}, \\ b - c &= \frac{1}{a} - \frac{1}{c} = \frac{c-a}{ca}, \\ c - a &= \frac{1}{b} - \frac{1}{a} = \frac{a-b}{ab}. \end{aligned}$$

Multiply all terms on both sides, we get

$$(a-b)(b-c)(c-a) = \frac{(b-c)(c-a)(a-b)}{(abc)^2}.$$

Because a, b, dan c are all distinct, $(a-b)(b-c)(c-a) \neq 0$, thus we are allowed to divide both sides by $(a-b)(b-c)(c-a)$. We infer that $abc = \pm 1$.
Hence, the largest possible value of abc is $\boxed{\textbf{E. 1}}$.

Bahasa Indonesia

Jawaban. 1

Solusi.
Sistem persamaan yang melibatkan simetri memiliki keunikan. Biasanya, kita dapat memanfaatkan sifat kesiklisan untuk menyusun persamaan(-persamaan) lain. Jika kita kurangkan ekspresi paling kiri dengan ekspresi di tengah, kurangkan ekspresi di tengah dengan ekspresi paling kanan, dan kurangkan ekspresi paling kanan dengan ekspresi paling kiri, kita memperoleh tiga persamaan:

$$\begin{aligned} a-b &= \frac{1}{c}-\frac{1}{b} = \frac{b-c}{bc}, \\ b-c &= \frac{1}{a}-\frac{1}{c} = \frac{c-a}{ca}, \\ c-a &= \frac{1}{b}-\frac{1}{a} = \frac{a-b}{ab}. \end{aligned}$$

Kalikan masing-masing ruas, didapat persamaan

$$(a-b)(b-c)(c-a) = \frac{(b-c)(c-a)(a-b)}{(abc)^2}.$$

Karena dikatakan a, b, dan c adalah bilangan berbeda, maka $(a-b)(b-c)(c-a) \neq 0$, sehingga kita dapat membagi kedua ruas dengan $(a-b)(b-c)(c-a)$. Diperoleh $abc = \pm 1$.
Maka, nilai terbesar yang mungkin dari abc adalah $\boxed{\textbf{E. 1}}$.

9. How many positive integers less that 2020 with the property that the sum of it digits equals 9?

A. 50
B. 100
C. 102
D. 202
E. None of the above

English

Answer. 102

Solution.
Let $n = \overline{abcd}$ be a positive integer less than 2020 and the sum of its digits is 9. We divide into cases according to the number in thousands' place.
Case 1. $a = 0$, we have

$$b + c + d = 9$$

where $0 \leq b, c, d \leq 9$.
Using *stars and bars* method, there are $\binom{9+3-1}{9} = 55$ possible solutions for this case.

Case 2. $a = 1$, we have

$$b + c + d = 8$$

where $0 \leq b, c, d \leq 9$.
Using *stars and bars* method, there are $\binom{8+3-1}{8} = 45$ possible solutions for this case.

Case 3. $a = 2$, only $n = 2007$ and $n = 2016$ satisfy the criteria.

In total, there are $55 + 45 + 2 = \boxed{\textbf{C. 102}}$ positive integers less than 2020 whose digits' sum is 9.

Bahasa Indonesia

Jawaban. 102

Solusi.
Misalkan $n = \overline{abcd}$ merupakan bilangan bulat positif yang lebih kecil daripada 2020 dan memiliki penjumlahan digit 9. Kita bagi kasus berdasarkan kemungkinan pada digit ribuan.

Kasus 1. $a = 0$, kita memiliki

$$b + c + d = 9$$

dimana $0 \leq b, c, d \leq 9$.

Berdasarkan metode *stars and bars*, terdapat $\binom{9+3-1}{9} = 55$ solusi pada kasus ini.

Kasus 2. $a = 1$, kita memiliki

$$b + c + d = 8$$

dimana $0 \leq b, c, d \leq 9$.

Berdasarkan metode *stars and bars*, terdapat $\binom{8+3-1}{8} = 45$ solusi pada kasus ini.

Kasus 3. $a = 2$, hanya 2007 dan 2016 yang memenuhi kondisi yang diminta.

Akhirnya, terdapat $55 + 45 + 2 =$ $\boxed{\textbf{C. 102}}$ bilangan bulat positif yang lebih kecil dari 2020 yang memiliki penjumlahan digit 9.

10. The sequence a_n is defined by

$$a_{n+2} = \frac{1 + a_{n+1}}{a_n}$$

with $a_1 = 1$ and $a_2 = 2$. Evaluate a_{2020}.

A. 1
B. 2
C. 3
D. 4
E. None of the above

English

Answer. 1

Solution.
For problems involving sequences, a recommended first step is to try out several first terms, from which we may find a hidden pattern or form hypothesis. We proceed for a few first terms of$\{a_n\}$:

$$a_1 = 1,\ a_2 = 2,\ \ a_3 = \frac{1+2}{1} = 3,$$
$$a_4 = \frac{1+3}{2} = 2,\ \ a_5 = \frac{1+2}{3} = 1,$$
$$a_6 = \frac{1+1}{2} = 1,\ \ a_7 = \frac{1+1}{1} = 2.$$

Notice the pattern. The way a_8 is dependent to a_6 and a_7 is the same as how a_3 is dependent to a_1 and a_2, and so on for $a_9, a_{10}, \cdots$.

We can rewrite the sequence a_n as:

$$a_n = \begin{cases} 1, & n = 5k+1; \\ 2, & n = 5k+2; \\ 3, & n = 5k+3; \\ 2, & n = 5k+4; \\ 1, & n = 5k+5. \end{cases}$$

for any nonnegative integers n and k. Hence, $a_{2020} = a_{5\cdot 403+5} = \boxed{\textbf{A. 1}}$.

Bahasa Indonesia

Jawaban. 1

Solusi.
Untuk soal barisan, langkah pertama yang dianjurkan adalah dengan mencoba beberapa bilangan pertama dari barisan. Dengan demikian, kita mungkin saja dapat melihat pola yang terbentuk, atau membuat hipotesis. Kita coba beberapa nilai awal dari $\{a_n\}$:

$$\begin{gathered} a_1 = 1,\ a_2 = 2,\ \ a_3 = \frac{1+2}{1} = 3, \\ a_4 = \frac{1+3}{2} = 2,\ \ a_5 = \frac{1+2}{3} = 1, \\ a_6 = \frac{1+1}{2} = 1,\ \ a_7 = \frac{1+1}{1} = 2. \end{gathered}$$

Perhatikan pola yang terbentuk. a_8 bergantung pada a_6 dan a_7 akan sama sebagaimana a_3 bergantung pada a_1 dan a_2, demikian seterusnya untuk $a_9, a_{10}, \cdots$.

Kita dapat tulis ulang barisan a_n sebagai berikut:

$$a_n = \begin{cases} 1, & n = 5k+1; \\ 2, & n = 5k+2; \\ 3, & n = 5k+3; \\ 2, & n = 5k+4; \\ 1, & n = 5k+5. \end{cases}$$

untuk sembarang bilangan bulat nonnegatif n dan k. Maka, $a_{2020} = a_{5\cdot 403+5} = \boxed{\textbf{A. 1}}$.

11. Find the smallest prime factor of

$$\underbrace{1000\cdots01}_{\text{2020 zeros}}$$

A. 3

B. 5

C. 7

D. 11

E. None of the above

English

Answer. 11

Solution.
Notice that

$$\underbrace{1000\cdots01}_{n \text{ zeroes}} = 10^{n+1} + 1$$

so our number can be written as

$$\underbrace{1000\cdots01}_{\text{2020 zeroes}} = 10^{2021} + 1.$$

We iterate starting from the smallest prime.

Checking divisibility by 2
Being an odd number, $10^{2021} + 1$ is not divisible by 2. Alternatively, as $10 \equiv 0 \pmod 2$ we have $10^{2021} + 1 \equiv 1 \pmod 3$.[a]

Checking divisibility by 3
The number is not divisible by 3 as the sum of its digits is not a number multiple of 3. Alternatively, as $10 \equiv 1 \pmod 3$ we have $10^{2021} + 1 \equiv 2 \pmod 3$.

Checking divisibility by 5
The number is not divisible by 5 as its last digit is neither 5 nor 0. Alternatively, as $10 \equiv 0 \pmod 5$ we have $10^{2021} + 1 \equiv 1 \pmod 5$.

Checking divisiblity by 7
As $10^2 \equiv -1 \pmod 7$, we have

$$\begin{aligned} 10^{2021} + 1 &\equiv 10 \cdot (10^2)^{1010} + 1 \pmod 7 \\ &\equiv 10 \cdot (-1)^{1010} + 1 \pmod 7 \\ &\equiv 5 \pmod 7. \end{aligned}$$

Checking divisibility by 11
We have $10 \equiv -1 \pmod{11}$, so

$$10^{2021} + 1 \equiv (-1)^{2021} + 1 \equiv 0 \pmod{11}.$$

We have found that $\boxed{\textbf{D. 11}}$ is the smallest prime dividing $\underbrace{1000\cdots01}_{\text{2020 zeros}}$.

[a] If a number n is divisible by p, then $n \equiv 0 \pmod p$.

Bahasa Indonesia

Jawaban. 11

Solusi.
Perhatikan bahwa

$$\underbrace{1000\cdots01}_{n \text{ nol}} = 10^{n+1}+1$$

sehingga

$$\underbrace{1000\cdots01}_{2020 \text{ nol}} = 10^{2021}+1.$$

Kita coba cek keterbagian dimulai dari prima terkecil.

Mengecek pembagian 2.
$10 \equiv 0 \pmod 2$, sehingga $10^{2021}+1 \equiv 1 \pmod 2$.[a] Alternatifnya, $10^{2021}+1$ merupakan bilangan ganjil sehingga tidak habis dibagi 2.

Mengecek pembagian 3.
$10 \equiv 1 \pmod 3$ sehingga $10^{2021}+1 \equiv 2 \pmod 3$. Alternatifnya, jumlah digit $10^{2021}+1$ bukan merupakan bilangan kelipatan tiga. Opsi A tidak mungkin.

Mengecek pembagian 5.
$10 \equiv 0 \pmod 5$ sehingga $10^{2021}+1 \equiv 1 \pmod 5$. Alternatifnya, digit terakhir $10^{2021}+1$ bukan 5 atau 0. Opsi B tidak mungkin.

Mengecek pembagian 7.
$10^2 \equiv -1 \pmod 7$ sehingga

$$\begin{aligned} 10^{2021}+1 &\equiv 10\cdot(10^2)^{1010}+1 \pmod 7\\ &\equiv 10\cdot(-1)^{1010}+1 \pmod 7\\ &\equiv 5 \pmod 7. \end{aligned}$$

Opsi C tidak mungkin.

Mengecek pembagian 11.
$10 \equiv -1 \pmod{11}$ sehingga

$$10^{2021}+1 \equiv (-1)^{2021}+1 \equiv 0 \pmod{11}.$$

Dengan demikian, **D. 11** merupakan faktor prima terkecil dari $\underbrace{1000\cdots01}_{2020 \text{ zeros}}$.

[a] Jika bilangan n habis dibagi p, maka $n \equiv 0 \pmod p$.

12. In the expansion of

$$f(x) = (1 + ax)^4(1 + bx)^5$$

where a and b are positive integers, the coefficient of x^2 is 86. Evaluate $a + b$.

A. 2
B. 3
C. 4
D. 5
E. None of the above

English

Answer. 3

Solution.
Use binomial expansion. Simply multiply terms that give terms containing x^2, which are

$$\begin{aligned}
\binom{4}{0}(1)^4(ax)^0 \times \binom{5}{2}(1)^3(bx)^2 &= (1 \times 10b^2)x^2, \\
\binom{4}{1}(1^3)(ax)^1 \times \binom{5}{1}(1^4)(bx)^1 &= (4a \times 5b)x^2, \\
\binom{4}{2}(1^2)(ax)^2 \times \binom{5}{0}(1^5)(bx)^0 &= (6a^2 \times 1)x^2.
\end{aligned}$$

Hence we obtain an equality, $6a^2 + 10b^2 + 20ab = 86$.

Unfortunately, we can't extract any other helpful equation from the known information, so we make an educated guess to the values of (a, b). Because the discrepancy of both sides' coefficient is not big, we guess for small values of (a, b).

Values (a, b) that satisfy the equation is $a = 1$ dan $b = 2$, so the value of $a + b$ is $\boxed{\textbf{B. 3}}$.

Bahasa Indonesia

Jawaban. 3

Solusi.
Gunakan ekspansi binomial. Cukup kalikan suku-suku yang akan menghasilkan x^2, yaitu

$$\binom{4}{0}(1)^4(ax)^0 \times \binom{5}{2}(1)^3(bx)^2 = (1 \times 10b^2)x^2,$$
$$\binom{4}{1}(1^3)(ax)^1 \times \binom{5}{1}(1^4)(bx)^1 = (4a \times 5b)x^2,$$
$$\binom{4}{2}(1^2)(ax)^2 \times \binom{5}{0}(1^5)(bx)^0 = (6a^2 \times 1)x^2.$$

Maka, kita peroleh persamaan $6a^2 + 10b^2 + 20ab = 86$.

Sayangnya, kita tidak dapat mengekstrak info lain yang dapat membantu menyelesaikan persoalan, sehingga kita coba menebak nilai (a, b) yang memenuhi. Karena koefisien pada ruas kiri dan angka pada ruas kanan tidak terlampau jauh, cukup coba nilai (a, b) yang kecil.

Pasangan (a, b) yang memenuhi adalah $a = 1$ dan $b = 2$, sehingga nilai $a + b$ adalah $\boxed{\textbf{B. 3}}$.

13. The equation $x^3 - ax^2 + bx - 2020$ has three positive integer roots.
Find the least possible value of a.

A. 101
B. 110
C. 202
D. 220
E. None of the above

English

Answer. 110

Solution.
Let x_1, x_2, dan x_3 be the roots of equation $x^3 - ax^2 + bx - 2020 = 0$. By Vieta's formula, we have

$$\begin{aligned} x_1 + x_2 + x_3 &= a, \\ x_1x_2x_3 &= 2020. \end{aligned}$$

As the roots are positive integers, we examine the prime factorization of 2020, which is $2^2 \times 5 \times 101$. We seek the least possible sum of the roots .

We list the possible triples of (x_1, x_2, x_3), excluding its permutations:

$$\{(4, 5, 101), (2, 10, 101), (2, 5, 202)\},$$

from which we could find that the minimum value of a is $4 + 5 + 101 = \boxed{\textbf{B. 110}}$.

Bahasa Indonesia

Jawaban. 110

Solusi.
Misalkan x_1, x_2, dan x_3 akar-akar persamaan $x^3 - ax^2 + bx - 2020 = 0$. Berdasarkan rumus Vieta, kita memiliki

$$\begin{aligned} x_1 + x_2 + x_3 &= a \\ x_1x_2x_3 &= 2020. \end{aligned}$$

Perhatikan bahwa $2020 = 2^2 \times 5 \times 101$.
Dengan mudah didapat bahwa nilai minimum a terjadi pada saat $x_1 = 2^2, x_2 = 5$, dan $x_3 = 101$.
Akhirnya $a = 2^2 + 5 + 101 = \boxed{\textbf{B. 110}}$.

14. Evaluate the sum

$$S = \sin^2 0° + \sin^2 2° + \sin^2 4° + \cdots + \sin^2 180°$$

.

A. 80
B. 81
C. 88
D. 90
E. None of the above

English

Answer. 45

Solution.
Relationship between the sine of an angle and the sine of its supplementary angle is

$$\sin x° = \sin(180° - x°).$$

With this in mind, if we regroup the terms in S as

$$S = (\sin^2 0° + \sin^2 180°) + (\sin^2 2° + \sin^2 178°) + \cdots + (\sin^2 88° + \sin^2 92°) + \sin^2 90°$$

and adding the terms inside each bracket, then S is equal to

$$S = 2 \times (\sin^2 0° + \sin^2 2° + \cdots + \sin^2 88°) + \sin^2 90°.$$

We add both sides by $\sin^2 90° = 1$. The reason for this will be apparent soon.

$$S + 1 = 2 \times (\sin^2 0° + \sin^2 2° + \cdots + \sin^2 88° + \sin^2 90°).$$

If we rearrange terms inside the bracket,

$$S + 1 = 2 \times [\underbrace{(\sin^2 0° + \sin^2 90°) + (\sin^2 2° + \sin^2 88°) + \cdots + (\sin^2 44° + \sin^2 46°)}_{\text{23 pairs}}]$$

and noting that

$$\begin{aligned} \sin^2 x° + \sin^2(90° - x°) &= \sin^2 x° + \cos^2 x° \\ &= 1, \end{aligned}$$

then $S + 1 = 2 \times (23 \times 1)$, from which we deduce that S is equal to $\boxed{\textbf{E. 45}}$.

Bahasa Indonesia

Jawaban. 45

Solusi.
Hubungan antara sinus sebuah sudut dengan sinus sudut suplemennya adalah

$$\sin^2 x^\circ = \sin^2(180^\circ - x^\circ).$$

Kita dapat menulis ulang S menjadi

$$S = (\sin^2 0^\circ + \sin^2 180^\circ) + (\sin^2 2^\circ + \sin^2 178^\circ) + \cdots + (\sin^2 88^\circ + \sin^2 92^\circ) + \sin^2 90^\circ.$$

Jika kita jumlahkan suku-suku pada tiap kurung, kita memperoleh

$$S = 2 \times (\sin^2 0^\circ + \sin^2 2^\circ + \cdots + \sin^2 88^\circ) + \sin^2 90^\circ.$$

Tambahkan kedua sisi dengan $\sin^2 90^\circ = 1$. Alasan hal ini dilakukan akan tampak pada langkah selanjutnya.

$$S + 1 = 2 \times (\sin^2 0^\circ + \sin^2 2^\circ + \cdots + \sin^2 88^\circ + \sin^2 90^\circ).$$

Jika kita susun ulang suku-suku di dalam kurung,

$$S + 1 = 2 \times [\underbrace{(\sin^2 0^\circ + \sin^2 90^\circ) + (\sin^2 2^\circ + \sin^2 88^\circ) + \cdots + (\sin^2 44^\circ + \sin^2 46^\circ)}_{\text{23 pasang}}]$$

dan mengingat bahwa

$$\begin{aligned} \sin^2 x^\circ + \sin^2(90^\circ - x^\circ) &= \sin^2 x^\circ + \cos^2 x^\circ \\ &= 1. \end{aligned}$$

maka $S + 1 = 2 \times (23 \times 1)$. Dari sana kita peroleh nilai S sama dengan $\boxed{\textbf{E. 45}}$.

15. Given that a and b are real numbers satsifying

$$\begin{cases} 6 - 5a + 4b - 3a^2 + 2ab - b^2 = 0, \\ a - b = 1. \end{cases}$$

Find the sum of all possible values of $\dfrac{30a}{b}$.

A. -15

B. -10

C. 15

D. 30

E. None of the above

English

Answer. -15

Solution.
Substitute $a = b + 1$ to the first equation:

$$\begin{aligned} 6 - 5(b+1) + 4b - 3(b+1)^2 + 2(b+1)b - b^2 &= 0 \\ \implies 2b^2 + 5b + 2 &= 0, \end{aligned}$$

whose solutions are $b = -2$ and $b = -\frac{1}{2}$.
If $b = -2$, then $a = -1$ so we have $\dfrac{30a}{b} = 15$.
If $b = -\frac{1}{2}$, then $a = \frac{1}{2}$ so we have $\dfrac{30a}{b} = -30$.
Hence, the sum of all possible value of $\dfrac{30a}{b}$ is $15 + (-30) = \boxed{\textbf{A. -15}}$

Bahasa Indonesia

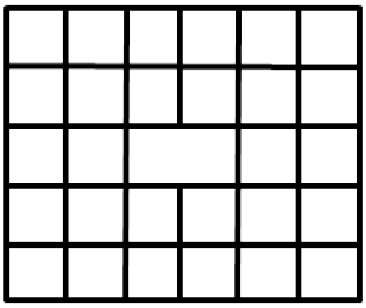
Jawaban. -15

Solusi.
Substitusi persamaan $a = b + 1$ ke persamaan pertama:

$$\begin{aligned} 6 - 5(b+1) + 4b - 3(b+1)^2 + 2(b+1)b - b^2 &= 0 \\ \implies 2b^2 + 5b + 2 &= 0 \end{aligned}$$

memiliki solusi $b = -2$ atau $b = -\frac{1}{2}$.
Jika $b = -2$, didapat $a = -1$ sehingga $\dfrac{30a}{b} = 15$.
Jika $b = -\frac{1}{2}$, didapat $a = \frac{1}{2}$ sehingga $\dfrac{30a}{b} = -30$.
Dengan demikian, penjumlahan semua nilai $\dfrac{30a}{b}$ yang mungkin adalah $15 + (-30) = \boxed{\textbf{A. -15}}$

16. The figure below shows a 5×6 rectangular board with a missing 1×2 rectangle in the center.



How many squares are there in the board?

A. 14

B. 30

C. 54

D. 56

E. None of the above

English

Answer. 56

Solution.
Suppose that, if the rectangle in the center weren't missing, we would have:
$6 \times 5 = 30$ squares of size 1×1;
$5 \times 4 = 20$ squares of size 2×2;
$4 \times 3 = 12$ squares of size 3×3;
$3 \times 2 = 6$ squares of size 4×4; and
$2 \times 1 = 2$ squares of size 5×5.
Due to it being actually missing, we lost 2 squares of size 1×1, 6 squares of size 2×2, and 6 squares of size 3×3.
Hence, the amount of squares in the board is $30 + 20 + 12 + 6 + 2 - 2 - 6 - 6 = \boxed{\textbf{D. 56}}$ squares.

Bahasa Indonesia

Answer. 56

Solution.
Andaikan persegi panjang di bagian tengah tidak hilang, maka kita punya:
$6 \times 5 = 30$ persegi ukuran 1×1;
$5 \times 4 = 20$ persegi ukuran 2×2;
$4 \times 3 = 12$ persegi ukuran 3×3;
$3 \times 2 = 6$ persegi ukuran 4×4; dan
$2 \times 1 = 2$ persegi ukuran 5×5.
Karena persegi panjang di tengah tidak ada, kita kehilangan 2 persegi ukuran 1×1, kehilangan persegi ukuran 2×2, dan kehilangan persegi ukuran 3×3.
Maka, banyaknya persegi pada papan tersebut adalah $30 + 20 + 12 + 6 + 2 - 2 - 6 - 6 = \boxed{\textbf{D. 56}}$ persegi.

17. In ΔABC,

$$(\sin A + \sin B) : (\sin B + \sin C) : (\sin C + \sin A) = 19\ : 20\ : 21.$$

Find the value of $99 \cos A$.

A. 39

B. 41

C. 51

D. 60

E. None of the above

English

Asnwer. 51

Solution.
Suppose that the quantity divided by its ratio is equal to some number k. We may rewrite the equality to

$$\frac{\sin A + \sin B}{19} = \frac{\sin B + \sin C}{20} = \frac{\sin C + \sin A}{21} = k$$
$$\implies (\sin A, \sin B, \sin C) = (10k, 9k, 11k).$$

If we apply the sine rule to ΔABC, we have

$$\frac{A}{\sin A} = \frac{B}{\sin B} = \frac{C}{\sin C} = m$$
$$\frac{A}{10k} = \frac{B}{9k} = \frac{C}{11k} = m$$
$$\implies (A, B, C) = (10mk, 9mk, 11mk)$$

for some number m.

To look for $\cos A$, we use the cosine rule:

$$\cos A = \frac{B^2 + C^2 - A^2}{2BC}$$
$$= \frac{(9mk)^2 + (11mk)^2 - (10mk)^2}{2(9mk)(11mk)} = \frac{17}{33}.$$

Thus, the value of $99 \cos A$ is $\boxed{\textbf{C. 51}}$.

Bahasa Indonesia

Jawaban. 51

Solusi.
Misalkan rasio perbandingan tiap kuantitas sama dengan k. Tulis ulang persamaan menjadi

$$\frac{\sin A+\sin B}{19}=\frac{\sin B+\sin C}{20}=\frac{\sin C+\sin A}{21}=k$$
$$\Longrightarrow (\sin A,\sin B,\sin C)=(10k,9k,11k).$$

Kita akan menggunakan aturan sinus pada ΔABC:

$$\frac{A}{\sin A}=\frac{B}{\sin B}=\frac{C}{\sin C}=m$$
$$\frac{A}{10k}=\frac{B}{9k}=\frac{C}{11k}=m$$
$$\Longrightarrow (A,B,C)=(10mk,9mk,11mk).$$

untuk sebuah nilai m.

Untuk mencari nilai $\cos A$, kita dapat menggunakan aturan kosinus:

$$\begin{aligned}\cos A&=\frac{B^2+C^2-A^2}{2BC}\\&=\frac{(9mk)^2+(11mk)^2-(10mk)^2}{2(9mk)(11mk)}=\frac{17}{33}.\end{aligned}$$

Akhirnya, nilai $99\cos A$ adalah $\boxed{\textbf{C. 51}}$.

18. Find the least positive integer n such that the equation $\left\lfloor \frac{10^n}{x} \right\rfloor = 98$ has integer solution x.

$\lfloor k \rfloor$ is the largest integer smaller than or equal to k.

A. 3
B. 4
C. 5
D. 6
E. None of the above

English

Answer. 4

Solution.
Fractional part of a non-negative real number x, denoted by $\{x\}$, is the remainder when x is subtracted by its integer part. This can be stated using the floor function:

$$\{x\} = x - \lfloor x \rfloor,$$

where $\lfloor x \rfloor$ returns the greatest integer less than or equal to x.

We have

$$\left\{ \frac{10^n}{x} \right\} = \frac{10^n}{x} - 98.$$

Based on the definition of fractional part, we have $0 \le \{x\} < 1$, thus

$$0 \le \frac{10^n}{x} - 98 < 1$$
$$\implies 98x \le 10^n < 99x.$$

The implication is obtained by multiplying each side by x, which is permitted as x is always positive.[a]

We iterate from the smallest positive integer.
For $n \le 3$, no integer x satisfies the above inequality.
For $n = 4$, $x = 102$. Hence, the least positive integer n that satisfies the required condition is $\boxed{\textbf{B. 4}}$.

[a] In case it is not obvious, notice that 10^n is always positive for every positive n. Thus, by dividing by a positive x, we obtain a positive number whose floor function value is also positive, which is not the case if x is not positive.

Bahasa Indonesia

Jawaban. 4

Solusi.
Perhatikan bahwa

$$\lfloor x \rfloor + \{x\} = x,$$

dimana $\lfloor x \rfloor$ merupakan bilangan bulat terkecil yang tidak lebih besar atau sama dengan x dan $\{x\}$ menyatakan bagian pecahan dari x.
Berdasarkan definisi tersebut, kita memiliki $0 \leq \{x\} < 1$ dan

$$\begin{aligned}
\frac{10^n}{x} - \left\{\frac{10^n}{x}\right\} &= 98 \\
\left\{\frac{10^n}{x}\right\} &= \frac{10^n}{x} - 98 \\
\implies 0 \leq \frac{10^n}{x} &- 98 < 1 \\
98x \leq 10^n &< 99x.
\end{aligned}$$

Jika $n = 3$, tidak ada nilai x bulat yang memenuhi pertidaksamaan tersebut.
Jika $n = 4$, $x = 102$. Jadi, bilangan bulat n terkecil yang memenuhi adalah $\boxed{\textbf{B. 4}}$.

19. How many positive integers $k < 100$ such that

$$2(5^{6n}) + k(2^{3n+2}) - 1$$

is divisible by 7 for any positive integer n?

A. 12
B. 14
C. 18
D. 19
E. None of the above

English

Answer. 14

Solution.
As with most other problems concerning divisibility, we have the modulo as our tool. In this case, notice that $5^3 \equiv -1 \pmod 7$ and $2^3 \equiv 1 \pmod 7$.

$$\begin{aligned} 2(5^{6n}) + k(2^{3n+2}) - 1 &\equiv 2(-1)^{2n} + k \cdot (1)^n \cdot 4 - 1 \pmod 7 \\ &\equiv 4k + 1 \pmod 7 \end{aligned}$$

In order for $2(5^{6n}) + k(2^{3n+2}) - 1$ to be divisible by 7, then $2(5^{6n}) + k(2^{3n+2}) - 1 \equiv 0 \pmod 7$.
As the number is also equivalent to $4k + 1 \pmod 7$, k must satisfy $k \equiv 5 \pmod 7$.
Thus, we have $k = 7c + 5$ for every positive integer c.
In order for $k < 100$, then $c = \{0, 1, 2, \cdots, 13\}$.
Hence, there are 14 values of c which correspond to **B. 14** distinct positive integers k such that $2(5^{6n}) + k(2^{3n+2}) - 1$ is divisible by 7 for any positive integer n.

Bahasa Indonesia

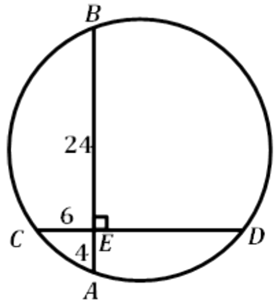
Jawaban. 14

Solusi.
Perhatikan bahwa $5^3 \equiv -1 \pmod 7$ dan $2^3 \equiv 1 \pmod 7$.

$$\begin{aligned} 2(5^{6n}) + k(2^{3n+2}) - 1 &\equiv 2(-1)^{2n} + k \cdot (1)^n \cdot 4 - 1 \pmod 7 \\ &\equiv 4k + 1 \pmod 7 \\ \implies k &\equiv 5 \pmod 7. \end{aligned}$$

Dengan demikian, kita mendapati $k = 7c + 5$ untuk setiap bilangan bulat c.
Agar $k < 100$, $c = \{0, 1, 2, \cdots, 13\}$.
Jadi, terdapat 14 nilai c yang memenuhi sehingga terdapat **B. 14** bilangan bulat positif k sedemikian sehingga $2(5^{6n}) + k(2^{3n+2}) - 1$ habis dibagi 7 untuk setiap bilangan bulat n.

20. A, B, C, and D are four distinct points lying on the circumference of a circle such that chords AB and CD are perpendicular at point E.



Given that $EA = 4$, $EB = 2$, and $EC = 6$, find the radius of the circle.

A. $\sqrt{221}$
B. 15
C. $\sqrt{270}$
D. 18
E. None of the above

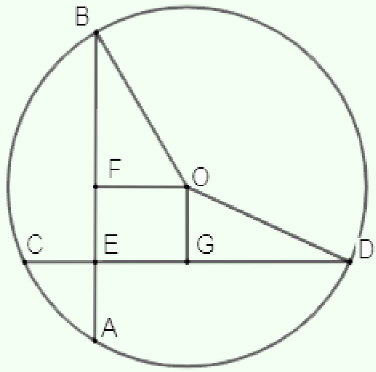
English

Answer. $\sqrt{221}$

Solution.
Denote O as the center of the circle. Draw a line from O such that it intersects $\overline{AB}$ at point F. Similarly, draw a line from O such that it intersects $\overline{CD}$ at point G, as shown in the diagram below.

According to *power of a point*, specifically intersecting chords theorem, we have the relationship

$$\begin{aligned} EC \times ED &= EB \times EA \\ 6 \times ED &= 24 \times 4 \\ ED &= 16. \end{aligned}$$



Notice that the two lines we have previously drawn are the circle's apothems. A property unique to apothems is that they perpendicularly bisect the chord into two line segments of equal length. With that in mind, we therefore have $BF = 14$ and $FO = EG = 5$.

We can find the circle's radius by applying Pythagorean theorem on ΔBFO:

$$BO = \sqrt{BF^2 + FO^2} = \sqrt{14^2 + 5^2} = \sqrt{221}.$$

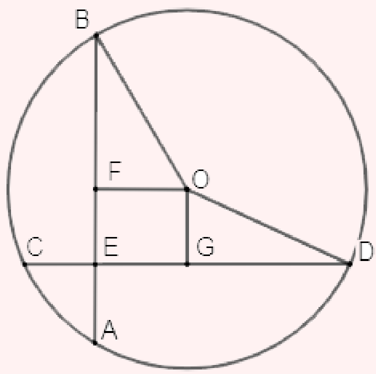
Hence the radius of the circle is $\boxed{\textbf{A. } \sqrt{221}}$.

Bahasa Indonesia

Jawaban. $\sqrt{221}$

Solusi.
Misalkan O sebagai titik pusat lingkaran. Misalkan pula F dan G berturut-turut sebagai titik tengah AB dan CD. Dengan power of a point:

$$\begin{aligned} EC \times ED &= EB \times EA \\ 6ED &= 24 \times 4 \\ \implies ED &= 16. \end{aligned}$$



Perhatikan bahwa $CG = GD$ dan $FB = FA$ sehingga $CG = 11, EG = 5, BF = 14$, dan $FE = 10$.
Dengan pythagoras:

$$\begin{aligned} BO &= \sqrt{BF^2 + FO^2} \\ &= \sqrt{BF^2 + EG^2} = \sqrt{14^2 + 5^2} = \sqrt{221}. \end{aligned}$$

Jadi, jari-jari persamaan lingkaran di atas adalah **A.** $\sqrt{221}$

21. You need to tile a 10×1 hallway with a supply of 1×1 red, 2×1 red tiles and 2×1 blue tiles. find the number of ways you can tile the 10×1 hallway.

English

Answer. 683

Solution.
We count the number of ways based on how many 2×1 is used.

If we plan not to use any 2×1 tile, the entire hallway will be tiled using 1×1 tiles. From this, we count that there is only one way.

If we plan to use one 2×1, then we are to use eight 1×1 tiles. Imagine that the 2×1 tile 'separates' the 8 1×1 tiles into two sections.[a] Using *stars and bars* method, we calculate the number of ways to tile:

$$\binom{8+2-1}{8} = 9 \text{ ways.}$$

Because we have two different colors, we have $9 \times 2^1 = 18$ ways to tile.

Similar argument can be applied to the rest of cases. For two 2×2 tiles, the possible number of tiling is (for this case, there are six 1×1 tiles separated into three sections)

$$\binom{6+3-1}{6} \times 2^2 = 112 \text{ ways.}$$

For three 2×2 tiles,

$$\binom{4+4-1}{4} \times 2^3 = 280 \text{ ways.}$$

For four 2×2 tiles,

$$\binom{2+5-1}{2} \times 2^4 = 240 \text{ ways.}$$

For five 2×2 tiles, then we will not be using any 1×1 tiles, so the number of ways to tile is determined solely by color selection: $2^5 = 32$ ways.

Hence, the number of ways to tile the hallway is $1 + 18 + 112 + 280 + 240 + 32 = \boxed{\mathbf{683}}$ ways.

[a] A 'section' can be empty without any 1×1 tiles; i.e. the 2×2 tile is put at the end of the hallway.

Bahasa Indonesia

Jawaban. 683

Solusi.
Kita hitung banyak cara memasang ubin berdasarkan banyaknya ubin 2×1.

Jika tidak memasang ubin 2×1, sepanjang lorong hanya dipasang dengan ubin merah 1×1, maka hanya terdapat satu cara.

Jika berencana memasang 1 ubin 2×1, maka kita akan menggunakan 8 ubin 1×1. Pandang ubin 2×1 sebagai 'penyekat' di ruang antara 8 ubin 1×1 sehingga terbagi menjadi 2 bagian.[a] Maka, banyaknya cara memasang ubin dapat dicari menggunakan metode *stars and bars*:

$$\binom{8+2-1}{8} = 9 \text{ ways.}$$

Karena terdapat dua warna berbeda, maka banyaknya cara pengubinan adalah $9 \times 2^1 = 18$ cara.

Argumen yang serupa dapat digunakan untuk kasus tersisa. Untuk 2 ubin 2×2, banyaknya cara memasang ubin adalah (untuk kasus ini, ada 6 ubin 1×1 yang dibagi menjadi 3 bagian)

$$\binom{6+3-1}{6} \times 2^2 = 112 \text{ cara.}$$

Untuk 3 ubin 2×2,

$$\binom{4+4-1}{4} \times 2^3 = 280 \text{ cara.}$$

Untuk 4 ubin 2×2,

$$\binom{2+5-1}{2} \times 2^4 = 240 \text{ cara.}$$

Untuk 5 ubin 2×2, tidak ada ubin 1×1, maka banyaknya cara ditentukan dari pilihan warna: $2^5 = 32$ cara.

Maka, banyaknya cara pengubinan seperti yang diminta adalah $1 + 18 + 112 + 280 + 240 + 32 = \boxed{\mathbf{683}}$ cara.

[a]Satu 'bagian' bisa saja 'kosong', artinya bisa aja ubin 2×2 dipasang di ujung lorong.

22. x, y, and z are real numbers such that

$$\begin{aligned} x + y + z &= 7 \\ x^2 + y^2 + z^2 &= 19 \\ x^3 + y^3 + z^3 &= 64. \end{aligned}$$

Evaluate $x^4 + y^4 + z^4$.

English

Answer. 247

Solution.
We manipulate the three equations above algebraically to obtain the desired expression. For the sake of explanation, we label the equations above:

$$x + y + z = 7, \tag{1}$$

$$x^2 + y^2 + z^2 = 19, \tag{2}$$

$$x^3 + y^3 + z^3 = 64. \tag{3}$$

If we square both sides of equation (1),

$$\begin{aligned} (x + y + z)^2 &= 7^2 \\ \underbrace{x^2 + y^2 + z^2}_{(2)} + 2(xy + yz + xz) &= 49, \end{aligned}$$

we obtain

$$xy + yz + xz = 15. \tag{4}$$

If we multiply equation (1) with equation (2),

$$\begin{aligned} (x + y + z)(x^2 + y^2 + z^2) &= 7 \times 19 \\ \underbrace{x^3 + y^3 + z^3}_{(3)} + x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2 &= 133 \end{aligned}$$

then we get

$$x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2 = 69. \tag{5}$$

If we take the third power to both sides of equation (1),

$$\begin{aligned} (x + y + z)^3 &= 7^3 \\ \underbrace{x^3 + y^3 + z^3}_{(3)} + 3(\underbrace{x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2}_{(5)}) + 6xyz &= 343 \end{aligned}$$

then we obtain

$$xyz = 12. \tag{6}$$

Continuing on the next page...

Squaring both sides of equation (4),

$$\begin{aligned}(xy+yz+xz)^2 &= 15^2\\ x^2y^2+y^2z^2+x^2z^2+2(xy^2z+x^2yz+xyz^2) &= 225\\ x^2y^2+y^2z^2+x^2z^2+2(\underbrace{xyz}_{(6)})(\underbrace{x+y+z}_{(1)}) &= 225\end{aligned}$$

then we obtain

$$x^2y^2+y^2z^2+x^2z^2 = 57. \tag{7}$$

Finally, to acquire the desired expression, we square both sides of equation (2):

$$\begin{aligned}(x^2+y^2+z^2)^2 &= 19^2\\ x^4+y^4+z^4+2(\underbrace{x^2y^2+y^2z^2+x^2z^2}_{(7)}) &= 361\end{aligned}$$

Consequently, the value of $x^4+y^4+z^4$ is $\boxed{\mathbf{247}}$.

Bahasa Indonesia

Jawaban. 247

Solusi.
Kita manipulasi ketiga persamaan di atas menjadi bentuk yang kita inginkan. Sebelumnya, kita perlu mencari beberapa bentuk lain yang dibutuhkan. Untuk memudahkan penjelasan, kita beri label persamaan di atas:

$$x + y + z = 7, \tag{1}$$

$$x^2 + y^2 + z^2 = 19, \tag{2}$$

$$x^3 + y^3 + z^3 = 64. \tag{3}$$

Jika kita kuadratkan kedua ruas pada persamaan (1),

$$\begin{aligned} (x+y+z)^2 &= 7^2 \\ \underbrace{x^2+y^2+z^2}_{(2)} + 2(xy+yz+xz) &= 49, \end{aligned}$$

kita peroleh

$$xy + yz + xz = 15. \tag{4}$$

Jika kita kalikan persamaan (1) dengan persamaan (2),

$$\begin{aligned} (x+y+z)(x^2+y^2+z^2) &= 7 \times 19 \\ \underbrace{x^3+y^3+z^3}_{(3)} + x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2 &= 133 \end{aligned}$$

maka kita mendapatkan

$$x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2 = 69. \tag{5}$$

Jika kita pangkatkan tiga kedua ruas pada persamaan (1),

$$\begin{aligned} (x+y+z)^3 &= 7^3 \\ \underbrace{x^3+y^3+z^3}_{(3)} + 3(\underbrace{x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2}_{(5)}) + 6xyz &= 343 \end{aligned}$$

kita dapat

$$xyz = 12. \tag{6}$$

Jika kita kuadratkan kedua ruas pada persamaan (4),

$$\begin{aligned} (xy+yz+xz)^2 &= 15^2 \\ x^2y^2 + y^2z^2 + x^2z^2 + 2(xy^2z + x^2yz + xyz^2) &= 225 \\ x^2y^2 + y^2z^2 + x^2z^2 + 2(\underbrace{xyz}_{(6)})(\underbrace{x+y+z}_{(1)}) &= 225 \end{aligned}$$

maka kita memperoleh

$$x^2y^2 + y^2z^2 + x^2z^2 = 57. \tag{7}$$

Bersambung ke halaman selanjutnya...

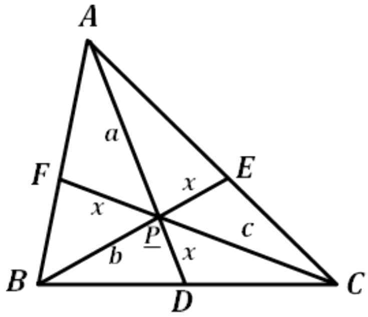
Untuk memperoleh bentuk yang diminta, kita kuadratkan kedua ruas pada persamaan (2):

$$(x^2 + y^2 + z^2)^2 = 19^2$$
$$x^4 + y^4 + z^4 + 2(\underbrace{x^2y^2 + y^2z^2 + x^2z^2}_{(7)}) = 361$$

Maka, nilai dari $x^4 + y^4 + z^4$ adalah $\boxed{\mathbf{247}}$.

23. In ΔABC shown below, AD, BE, and CF intersect at P. Suppose $AP = a, BP = b, CP = c$ and $DP = EP = FP = x$.

Given that $x = 3$ and $a + b + c = 20$, find abc.



English

Answer. 234

Solution.
Using the fact that ΔPBC and ΔABC share a same side in $\overline{BC}$, we may express the area ratio of the two triangles as

$$\frac{[\Delta PBC]}{[\Delta ABC]} = \frac{x}{a+x}.$$

Going by similar argument[a] to the fact that ΔPCA and ΔPAB share a side with ΔABC respectively, we obtain

$$\frac{[\Delta PCA]}{[\Delta ABC]} = \frac{x}{b+x} \quad \text{and} \quad \frac{[\Delta PAB]}{[\Delta ABC]} = \frac{x}{c+x}.$$

Therefore,

$$\begin{aligned}\frac{x}{a+x} + \frac{x}{b+x} + \frac{x}{c+x} &= \frac{[\Delta PBC]}{[\Delta ABC]} + \frac{[\Delta PCA]}{[\Delta ABC]} + \frac{[\Delta PAB]}{[\Delta ABC]} \\ &= \frac{[\Delta ABC]}{[\Delta ABC]} = 1.\end{aligned}$$

When $x = 3$,

$$\begin{aligned}\frac{3}{a+3} + \frac{3}{b+3} + \frac{3}{c+3} &= 1 \\ 3\left[(a+3)(b+3) + (b+3)(c+3) + (c+3)(a+3)\right] &= (a+3)(b+3)(c+3) \\ 3(ab+bc+ca) + 18(a+b+c) + 81 &= abc + 3(ab+bc+ca) + 9(a+b+c) + 27 \\ abc &= 9(a+b+c) + 54.\end{aligned}$$

Given that $a + b + c = 20$, the value of abc is $9(20) + 54 = \boxed{\mathbf{234}}$.

[a]In case the prior argument is not obvious, we draw two lines from both vertices perpendicular to the shared line segment. Then,

$$\frac{[\Delta PBC]}{[\Delta ABC]} = \frac{\frac{1}{2} \times P'P \times BC}{\frac{1}{2} \times A'A \times BC} = \frac{x \sin \angle PDB}{(a+x) \sin \angle ADB} = \frac{x}{a+x}.$$

Bahasa Indonesia

Jawaban. 234

Solusi.
Berdasarkan perbandingan luas, kita memiliki

$$\frac{x}{a+x} = \frac{\Delta PBC}{\Delta ABC}, \ \frac{x}{b+x} = \frac{\Delta PCA}{\Delta ABC}, \ \frac{x}{c+x} = \frac{\Delta PAB}{\Delta ABC}.$$

Dengan demikian,

$$\begin{aligned}\frac{x}{a+x} + \frac{x}{b+x} + \frac{x}{c+x} &= \frac{\Delta PBC}{\Delta ABC} + \frac{\Delta PCA}{\Delta ABC} + \frac{\Delta PAB}{\Delta ABC} \\ &= \frac{\Delta ABC}{\Delta ABC} = 1.\end{aligned}$$

Untuk $x = 3$,

$$\begin{aligned}\frac{3}{a+3} + \frac{3}{b+3} + \frac{3}{c+3} &= 1 \\ 3\left[(a+3)(b+3) + (b+3)(c+3) + (c+3)(a+3)\right] &= (a+3)(b+3)(c+3) \\ 3(ab+bc+ca) + 18(a+b+c) + 81 &= abc + 3(ab+bc+ca) + 9(a+b+c) + 27 \\ abc &= 9(a+b+c) + 54.\end{aligned}$$

Mengingat bahwa $a + b + c = 20$, nilai abc adalah $9(20) + 54 = \boxed{\mathbf{234}}$.

24. Positive integers a, b, and c are randomly selected from the set $\{1, 2, 3, \cdots, 2020\}$ with replacement. Find the probablility that $abc + ab + 2a$ is divisible by 5.

English

Answer. $\frac{41}{125}$

Solution.
Note that $abc + ab + 2a$ has a as one of the factors. If a is divisible by 5, then we are done. The probability of a is divisible by 5 is $\frac{1}{5}$.

What if a is not divisible by 5? We have the equivalent expression $a(bc + b + 2)$. If we want this expression to be divisible by 5, then $(bc + b + 2)$ must be divisible by 5.

We need to find conditions of b, c such that $b(c + 1) \equiv -2 \pmod 5 \equiv 3 \pmod 5$.

If we iterate all b and c in modulo 5, then the conditions for b and c in order for $(bc + b + 2)$ to be divisible by 5 are

$$\begin{aligned} b \equiv 1 \pmod 5 &\text{ and } c \equiv 2 \pmod 5, \\ b \equiv 2 \pmod 5 &\text{ and } c \equiv 3 \pmod 5, \\ b \equiv 3 \pmod 5 &\text{ and } c \equiv 0 \pmod 5, \\ b \equiv 4 \pmod 5 &\text{ and } c \equiv 1 \pmod 5. \end{aligned}$$

There are four pairs (b, c) out of all possible 25 pairs. The probability of picking triples (a, b, c) satisfying these criteria is $\frac{4}{5} \times \frac{4}{25} = \frac{16}{125}$.

Therefore, the probability that $abc + ab + 2a$ is divisible by 5 is $\frac{1}{5} + \frac{16}{125} = \boxed{\frac{41}{125}}$.

Bahasa Indonesia

Jawaban. $\dfrac{41}{125}$

Solusi.
Perhatikan bahwa bentuk $abc + ab + 2a$ memiliki faktor a. Jika a habis dibagi 5, maka kita selesai. Peluang untuk a habis dibagi 5 adalah $\frac{1}{5}$.

Bagaimana jika a tidak habis dibagi 5? Kita punya bentuk yang ekuivalen $a(bc + b + 2)$. Jika ekspresi ini habis dibagi 5, maka $(bc + b + 2)$ haruslah habis dibagi 5.

Kita perlu cari kriteria b, c sedemikian sehingga $b(c + 1) \equiv -2 \pmod 5 \equiv 3 \pmod 5$.

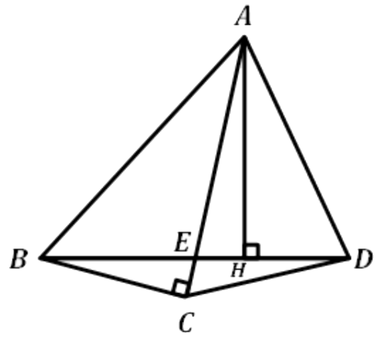
Jika kita iterasi semua kemungkinan b dan c dalam modulo 5, maka kriteria b, c yang memenuhi adalah

$$\begin{aligned} b &\equiv 1 \pmod 5 \text{ dan } c \equiv 2 \pmod 5, \\ b &\equiv 2 \pmod 5 \text{ dan } c \equiv 3 \pmod 5, \\ b &\equiv 3 \pmod 5 \text{ dan } c \equiv 0 \pmod 5, \\ b &\equiv 4 \pmod 5 \text{ dan } c \equiv 1 \pmod 5. \end{aligned}$$

Terdapat 4 pasangan (b, c) yang memenuhi kriteria dari 25 pasangan yang dapat dibentuk. Peluang terpilihnya (a, b, c) dengan kriteria ini adalah $\frac{4}{5} \times \frac{4}{25} = \frac{16}{125}$.

Maka, peluang bentuk $abc + ab + 2a$ habis dibagi 5 adalah $\dfrac{1}{5} + \dfrac{16}{125} = \boxed{\dfrac{41}{125}}$.

25. $ABCD$ is a convex quadrilateral such that AC intersects BD at E. H is a point lying in the segment DE such that AH is perpendicular to DE.
Suppose $BE = ED$, $CE = 9, EH = 12, AH = 32$, and $\angle BCA = 90°$. Evaluate the length of CD.



English

Answer. 30

Solution.
Using Pythagorean theorem on ΔEAH,

$$EA = \sqrt{AH^2 + EH^2}$$
$$= \sqrt{32^2 + 12^2} = 4\sqrt{73}.$$

Notice that $AHCB$ is a cyclic quadrilateral as we have $\angle BCA = \angle BHA$.[a] We apply *power of a point*, specifically intersecting chords theorem,

$$EC \times EA = EH \times EB$$
$$9(4\sqrt{73}) = 12 \times ED \ \ \text{(because } EB = ED\text{)}$$
$$\implies ED = 3\sqrt{73}.$$

Let $\angle HEA = \alpha$. On ΔHEA, $\cos\alpha = \frac{3}{\sqrt{73}}$.

Note that $\cos(\angle CED) = \cos(\pi - \alpha) = -\cos\alpha = -\frac{3}{\sqrt{73}}$.

Therefore, the length of $\overline{CD}$ can be determined using the law of cosines on ΔCED:

$$CD^2 = EC^2 + ED^2 - 2 \cdot EC \cdot ED \cdot \cos(\angle CED)$$
$$CD^2 = 9^2 + (3\sqrt{73})^2 - 2 \cdot 9 \cdot 3\sqrt{73} \cdot \frac{-3}{\sqrt{73}}$$
$$\implies CD = 30.$$

The length of $\overline{CD}$ is $\boxed{\mathbf{30}}$.

[a] Imagine a circle with ΔABC and ΔAHB inscribed into the circle. $\overline{AB}$ would be the diameter of the circle.

Bahasa Indonesia

Jawaban. 30

Solusi.
Dengan pythagoras,

$$\begin{aligned} EA &= \sqrt{AH^2 + EH^2} \\ &= \sqrt{32^2 + 12^2} = 4\sqrt{73}. \end{aligned}$$

Perhatikan bahwa $AHCB$ merupakan segiempat talibusur karena $\angle BCA = \angle BHA$. Dengan demikian, kita dapat menerapkan *power of a point*:

$$\begin{aligned} EC \times EA &= EH \times EB \\ 9(4\sqrt{73}) &= 12 \times ED \quad (\text{karena } EB = ED) \\ ED &= 3\sqrt{73}. \end{aligned}$$

Misalkan $\angle HEA = \alpha$, pada ΔHEA, $\cos\alpha = \dfrac{3}{\sqrt{73}}$.
Perhatikan pula bahwa $\cos(\angle CED) = \cos(\pi - \alpha) = -\cos\alpha = -\dfrac{3}{\sqrt{73}}$.
Sisi CD dapat ditentukan melalui aturan cos pada segitiga CED:

$$\begin{aligned} CD^2 &= EC^2 + ED^2 - 2 \cdot EC \cdot ED \cdot \cos(\angle CED) \\ CD^2 &= 9^2 + (3\sqrt{73})^2 - 2 \cdot 9 \cdot 3\sqrt{73} \cdot \frac{-3}{\sqrt{73}} \\ \implies CD &= 30. \end{aligned}$$

Jadi, panjang CD adalah $\boxed{\textbf{30}}$.